Exam "Numerieke Wiskunde", part I
Monday October 25, 2010, 9-12 hours

- It is *not* allowed to use the book or any notes
- Results from a previous item can be used, even if you didn't succeed to prove that item.
- **Good luck!**

1. We are going to construct adapted quadrature rules for integrals of type

$$I(f) = \int_0^1 f(x)\ln(x)dx$$

   (a) Why can't you expect too much from the application of a standard quadrature rule $g \mapsto \sum_i w_i g(x_i)$ for the approximation of $\int_0^1 g(x)dx$.

   (b) Show that for all $k \in \mathbb{N}_0$, $\int_0^1 x^k \ln(x)dx = -\frac{1}{(k+1)^2}$.

   (c) Let ϕ be the linear interpolation polynomial of f with interpolation points 0 and 1. Compute $\int_0^1 \phi(x)\ln(x)dx$.

   (d) Show that the error in this quadrature rule can be written as $cf^{(n)}(\xi)$ for some $\xi \in [0,1]$, and determine the constants c and n.

   (e) For $a \in (0,1]$, we now consider quadrature formulas of type $Q_a(f) = w_a f(a)$. Determine a and w_a such that $Q_a(p) = I(p)$ for all $p \in \mathcal{P}_1$.

   (f) Answer question (1d) for the quadrature formula from (1e).

2. Let $T_n(x) = \cos(n \arccos x)$. Obviously $\|T_n\|_\infty := \sup_{x\in[-1,1]} |T_n(x)| = 1$. Furthermore it is known that T_{n+1} has zeros $\cos(\frac{(j-\frac{1}{2})\pi}{n+1})$, $j = 1, \ldots n+1$; $T_{n+1} = \pm 1$ alternating in the points $\cos(\frac{j\pi}{n+1})$, $j = 0, \ldots, n+1$; and that $2^{-n}T_{n+1} \in \mathcal{P}^{(1)}_{n+1}$, being the set of polynomials of degree $n+1$ with leading coefficient 1.

   (a) For p_n being the Lagrange interpolation polynomial of degree n with interpolation points being the zeros of T_{n+1}, prove that, assuming a sufficiently smooth f,

$$\|f - p_n\|_\infty \leq \frac{2^{-n}}{(n+1)!} \|f^{(n+1)}\|_\infty.$$

   (b) Show that if $f^{(n+1)} > 0$ on $[-1, 1]$, then

$$\min_{q_n \in \mathcal{P}_n} \|f - q_n\|_\infty \geq \frac{2^{-n}}{(n+1)!} \min_{y\in[-1,1]} f^{(n+1)}(y).$$

   (c) Give a direct proof, so without applying a theorem from the book, of $2^{-n}T_{n+1} = \operatorname{argmin}_{p\in\mathcal{P}^{(1)}_{n+1}} \|p\|_\infty$. (Hint: Suppose not, and let $p \in \mathcal{P}^{(1)}_{n+1}$ with $\|p\|_\infty < \|f\|_\infty$. Consider $f - p$).

3. We consider $C[a, b]$ with inner product $\langle f, g\rangle = \int_a^b w(x)f(x)g(x)dx$ where w is a positive, continuous and integrable weight function on (a, b).

   (a) For $n \in \mathbb{N}_0$, show that there exists at most one $q_n \in \mathcal{P}^{(1)}_n$ with, for $n > 0$, $q_n \perp P_{n-1}$.

   (b) For such $(q_n)_{n\in\mathbb{N}_0}$, show that for any $n \in \mathbb{N}_0$, $\{q_0, \ldots, q_n\}$ is an orthogonal basis for $\mathcal{P}_n$.

   (c) Show that $(q_n)_{n\in\mathbb{N}_0}$ is determined by $q_0 := 1$, $q_1 := x - \frac{\langle x,q_0\rangle}{\langle q_0,q_0\rangle}$, and

$$q_{n+1} := \Big(x - \frac{\langle xq_n, q_n\rangle}{\langle q_n, q_n\rangle}\Big)q_n - \frac{\langle xq_n, q_{n-1}\rangle}{\langle q_{n-1}, q_{n-1}\rangle}q_{n-1} \quad (n \in \mathbb{N}).$$

   (d) Give a formula for the best approximation from $\mathcal{P}_n$ of $f \in C[a, b]$ with respect to $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.